Instituto Superior de Economia e Gestão
UNIVERSIDADE TÉCNICA DE LISBOA
DESDE 1911

Master in Actuarial Science

Loss Reserving

07-07-2017

Time allowed: 2 hours

Instructions:

1. This paper contains **13** questions and comprises **3** pages including the title page.
2. Enter all requested details on the cover sheet.
3. You must not start writing your answers until instructed to do so.
4. Number the pages of the paper where you are going to write your answers.
5. Attempt all questions.
6. Begin your answer to each question on a new page.
7. Marks are shown in brackets. Total marks: 200.
8. Show calculations where appropriate.
9. An approved calculator may be used.
10. Mobile phones and smartphones may not be used during the examination.

We have observed claims (counts or payments) for accident years $j = 1, \cdots, J$, where J denotes the current year. The set of observed, incremental claims is $\{X_{je} : j = 1, \cdots, J, e = 0, \cdots, J - j\}$, where X_{je} denotes claims from accident year j that emerge in calendar year $j + e$. The amount of risk exposed in accident year *j* has been p_j. The expected ultimate claim rate relative to the amount of risk we denote by θ_j and the proportion of claims emerging after *e* years, we denote by π_e.

Let us for argument's sake assume that the X_{je} are stochastically independent and follow a Poisson distribution with expected value $\mathrm{E}(X_{je}) = p_j \cdot \theta_j \cdot \pi_e$.

1. Prove that the maximum likelihood estimators of θ_j and π_e must satisfy the equations

$$X_{j,\le J-j} = p_j \cdot \theta_j^* \cdot \pi_{\le J-j}^* \qquad \text{for } j = 1, \cdots, J \text{, and}$$

$$X_{\le J-e,e} = \left(\sum_{j=1}^{J-e} p_j \cdot \theta_j^* \right) \cdot \pi_e^* \quad \text{for } e = 0, \cdots, J-1 .$$

[25 marks]

In this case, the likelihood of the observed data is

$$L = \prod_{j=1}^{J} \prod_{e=0}^{J-j} \frac{(p_j \theta_j \pi_e)^{X_{je}}}{X_{je}!} e^{-p_j \theta_j \pi_e}$$

Differentiating the log-likelihood and equating the derivatives to zero, an rearranging terms, we find the defining equations of the ML estimates:

$$X_{j,\le J-j} = p_j \theta_j^* \pi_{\le J-j}^* \text{ for } j = 1, \cdots, J \text{ and}$$

$$X_{\le J-e,e} = \left(\sum_{j=1}^{J-e} p_j \theta_j^* \right) \pi_e^* \text{ for } e = 0, \cdots, J-1 .$$

The equations in (1) cannot be solved directly and explicitly, but require a recursion.
There is a simpler way. Define development factors $\delta_e^* = \pi_{\le e}^* / \pi_{\le e-1}^*$ and $\delta_0^* = 1$.

2. Prove that $\pi_{\le e}^* = \prod_{e'=0}^{e} \delta_{e'}^* / \prod_{e'=0}^{J-1} \delta_{e'}^*$ [10 marks]

Start with the right hand side:

$$\frac{\prod_{e'=0}^{e}\delta_{e'}^*}{\prod_{e'=0}^{J-1}\delta_{e'}^*} = \frac{1}{\delta_{e+1}^*\delta_{e+2}^*\cdots\delta_{J-1}^*} = \frac{1}{\left(\frac{\pi_{\le e+1}^*}{\pi_{\le e}^*}\right)\left(\frac{\pi_{\le e+2}^*}{\pi_{\le e+1}^*}\right)\cdots\left(\frac{\pi_{\le J-1}^*}{\pi_{\le J-2}^*}\right)} = \frac{\pi_{\le e}^*}{\pi_{\le J-1}^*} = \pi_{\le e}^*$$ because $\pi_{\le J-1}^* = 1$

3. Write down how you can calculate the maximum likelihood estimates θ_j^* and π_e^*, starting with the empirical development factors $\delta_e^* = \sum_{j\le e} X_{j,\le e} / \sum_{j\le e} X_{j,\le e-1}$.

[20 marks]

The algorithm goes as follows:

a) First, calculate the empirical development factors $\delta_1^*,\cdots,\delta_{J-1}^*$ by $\delta_e^* = \sum_{j\le e} X_{j,\le e} / \sum_{j\le e} X_{j,\le e-1}$. Set $\delta_0^* = 1$.

b) Then calculate the accumulated development factors $\Delta_e^* = \prod_{e'=0}^{e}\delta_{e'}^*$ for $e = 0,\cdots,J-1$.

c) Calculate $\pi_{\le e}^* = \Delta_e^* / \Delta_{J-1}^*$ for $e = 0,\cdots,J-1$.

d) Calculate $\pi_e^* = \pi_{\le e}^* - \pi_{\le e-1}^*$ for $e = 1,\cdots,J-1$ and $\pi_0^* = \pi_{\le 0}^*$.

e) Calculate $\theta_j^* = X_{j,\le J-j} / p_j \pi_{\le J-j}^*$ for $j = 1,\cdots,J$.

This algorithm is straightforward to implement in a spreadsheet program.

4. Calculate the estimates $\{\theta_j^* : j = 1,\cdots,4\}$ and $\{\pi_e^* : e = 0,\cdots,3\}$ from the following data:

Accident year	Exposure
1	5124
2	4719
3	3898
4	3575

Claim counts	Development year			
Accident year	0	1	2	3
1	87	58	8	3
2	77	45	8	
3	79	41		
4	84			

[25 marks]

Empirical	Development year				
	0	1	2	3	4
Average		159.26 %	105.99 %	101.96 %	#DIV/0!
Selected		**159.26 %**	**105.99 %**	**101.96 %**	**100.00 %**

Convert delta to pi	Development year				
	0	1	2	3	4
delta (incr.)		1.5926	1.0599	1.0196	1.0000
delta (cum.)	100 %	159 %	169 %	172 %	172 %
pi (cum.)	58.10 %	92.53 %	98.08 %	100.00 %	100.00 %
pi (incr.)	**58.10 %**	**34.43 %**	**5.54 %**	**1.92 %**	**0.00 %**

Observed claim statistics and predicted future development

Accident year	Exposure	Developed to	Observed	pi(cum.)	**Theta(CL)**
1	5124	3	156	100 %	**3.04 %**
2	4719	2	130	98 %	**2.81 %**
3	3898	1	120	93 %	**3.33 %**
4	3575	0	84	58 %	**4.04 %**
Total	17316		490		3.17 %

5. Use the chain ladder method to predict the future claims. [10 marks]

Observed claim statistics and predicted future development

Accident year	Exposure	Developed to	Observed	pi(cum.)	Theta(CL)	Outstanding	Ultimate
1	5124	3	156	100 %	3.04 %	0	156
2	4719	2	130	98 %	2.81 %	3	133
3	3898	1	120	93 %	3.33 %	10	130
4	3575	0	84	58 %	4.04 %	61	145
Total	17316		490		**3.17 %**	73	563

6. Use the Cape Cod method to calculate a weighted average of $\{\theta_j^* : j = 1, \cdots, 4\}$ and denote it by θ^*. [10 marks]

$$\theta^* = \frac{\sum_j p_j \pi_{\le J-j} \theta_j^*}{\sum_j p_j \pi_{\le J-j}} = \frac{\sum_j X_{j, \le J-1}}{\sum_j p_j \pi_{\le J-j}} = \frac{\text{Observed claims}}{\text{"Observed exposure"}} = 3.17\%$$

Benktander's method is characterised by the prediction $\overline{X}_{j,>J-j} = p_j \overline{\theta}_j \pi_{>J-j}$, where $\overline{\theta}_j = \pi_{\le J-j}\theta_j^* + (1-\pi_{\le J-j})\theta^*$ is a weighted average.

7. Complete the following table, using Benktander's method.

j	p_j	$N_{j,\le J-j}$	$\pi_{\le J-j}$	θ_j^*	$\bar{\theta}_j$	$\bar{X}_{j,>J-j}$
1	5 124	156	100 %			
2	4 719	130	98 %			
3	3 898	120	93 %			
4	3 575	84	58 %			
Total	17 316	490				

[20 marks]

Accident year	Exposure	Observed	pi(cum.)	Theta(CL)	Theta_(Benkt.)	Outstanding
1	5124	156	100 %	3.04 %	3.04 %	0
2	4719	130	98 %	2.81 %	2.82 %	3
3	3898	120	93 %	3.33 %	3.32 %	10
4	3575	84	58 %	4.04 %	3.68 %	55
Total	17316	490		3.17 %		67

8. Explain the advantages of Benktander's method, compared with the chain ladder method. [10 marks]

More robust, less sensitive for immature years.

9. Describe some methods that you can use to smooth or extrapolate the development pattern in the tail, where data is scarce or unavailable. [10 marks]

Exponential decay, Weibull, many functions

10. Explain the meaning of the acronyms CBNI, IBNR and RBNS, then explain why "IBNR has more in common with CBNI than with RBNS". [20 marks]

Covered but not incurred
Incurred but not reported
Reported but not settled

CBNI and IBNR have in common that they're unreported. We don't know how many, we don't know how severe. On the other hand, of RBNS we know – potentially – a great deal.

Generalised linear models (GLM) can be used to model many different structures.

11. Describe the three components that define a GLM. [10 marks]

Covariate structure
Link function/response function
Probability distribution

12. Using GLM, propose a (simple!) joint model of two insurance portfolios. The portfolios are motor insurance in two different countries or regions. You assume that they have the same development pattern but different claim rates. You want to use the total statistical information from the two portfolios to estimate the development pattern and the two claim rates. [20 marks]

Model

$$\mathrm{E}\left(X_{je}^{(k)}\right) = p_j \cdot \exp\left(\beta_e + \gamma_k\right)$$

Where *k* is the country/region.

13. Explain the meaning of the assertion "Every claim cohort is a different portfolio when you use the chain ladder method". [10 marks]

The model in the previous question is very similar to the model underlying the chain ladder estimates. Every country is like a cohort.

END