Master in Actuarial Science

Loss Reserving

19-06-2017

Time allowed: 2 hours

Instructions:

1. This paper contains **11** questions and comprises **3** pages including the title page.
2. Enter all requested details on the cover sheet.
3. You must not start writing your answers until instructed to do so.
4. Number the pages of the paper where you are going to write your answers.
5. Attempt all questions.
6. Begin your answer to each question on a new page.
7. Marks are shown in brackets. Total marks: 200.
8. Show calculations where appropriate.
9. An approved calculator may be used.
10. Mobile phones and smartphones may not be used during the examination.

We have observed claim counts for accident years $j = 1, \cdots, J$, where J denotes the current year. The set of observed, incremental claim counts is $\{N_{jd} : j = 1, \cdots, J, d = 0, \cdots, J - j\}$, where N_{jd} denotes the number of claims from accident year j that are reported in calendar year $j + d$. The amount of risk exposed in accident year *j* has been p_j. The expected ultimate claim rate relative to the amount of risk we denote by θ, and the probability of a notification delay of *d* years, we denote by π_d. We assume that the N_{jd} are stochastically independent and follow a Poisson distribution with expected value $\mathrm{E}(N_{jd}) = p_j \cdot \theta \cdot \pi_d$.

1. Prove that the maximum likelihood estimators of θ and π_d are

$$\theta^* = \sum_{d=0}^{D} \left(\frac{N_{\le J-d,d}}{p_{\le J-d}} \right) \text{ and } \pi_d^* = \frac{N_{\le J-d,d}}{p_{\le J-d} \cdot \theta^*}.$$ [20 marks]

To estimate the claim frequencies and the delay probabilities one can maximise the likelihood of the observations:

(3.3) $$L = \prod_{j=1}^{J} \prod_{d=0}^{J-j} \frac{(p_j \theta \pi_d)^{N_{jd}}}{N_{jd}!} e^{-p_j \theta \pi_d} \quad \propto \quad \theta^{\sum_{d=0}^{D} N_{\le J-d,d}} \cdot \prod_{d=0}^{D} \pi_d^{N_{\le J-d,d}} \cdot \prod_{d=0}^{D} e^{-p_{\le J-d} \theta \pi_d}$$

Differentiating the log-likelihood one obtains

(3.4) $$\frac{\partial \ln L}{\partial \theta} = \frac{\sum_{d=0}^{D} N_{\le J-d,d}}{\theta} - \sum_{d=0}^{D} p_{\le J-d} \pi_d$$

and for *d*=0,…,*D*,

(3.5) $$\frac{\partial \ln L}{\partial \pi_d} = \frac{\sum_{d=0}^{D} N_{\le J-d,d}}{\pi_d} - \theta \cdot p_{\le J-d}.$$

Equating these expression to zero one obtains the defining equations of the ML estimates:

(3.6) $$\sum_{d=0}^{D} N_{\le J-d,d} = \theta^* \sum_{d=0}^{D} p_{\le J-d} \cdot \pi_d^*$$

and for *d*=0,…,*D*,

(3.7) $$N_{\le J-d,d} = \theta^* p_{\le J-d} \cdot \pi_d^*.$$

From this and the constraint $\sum_{d=0}^{D} \pi_d = 1$ one can derive the maximum likelihood estimates:

(3.8) $$\theta^* = \sum_{d=0}^{D} \left(\frac{N_{\le J-d,d}}{p_{\le J-d}} \right)$$

and, for d=0,…,D,

(3.9) $$\pi_d^* = \frac{N_{\le J-d,d}}{p_{\le J-d} \cdot \theta^*}.$$

2. Calculate the estimates θ^* and $\{\pi_d^* : d = 0,\cdots,3\}$ from the following data:

Accident year	Exposure
1	5124
2	4719
3	3898
4	3575

Claim counts	Development year			
Accident year	0	1	2	3
1	87	58	8	3
2	77	45	8	
3	79	41		
4	84			

[20 marks]

Exposure

Accident year	Exposure
1	5124
2	4719
3	3898
4	3575

Incremental claim statistics

Claim counts	Development year			
Accident year	0	1	2	3
1	87	58	8	3
2	77	45	8	
3	79	41		
4	84			

	Development year			
Empirical	0	1	2	3
Average	1.89 %	1.05 %	0.16 %	0.06 %
Selected	**1.89 %**	**1.05 %**	**0.16 %**	**0.06 %**

Theta	3.16 %

	Development year			
Delay pattern	0	1	2	3
pi (cum.)	59.81 %	93.00 %	98.15 %	100.00 %
pi (incr.)	59.81 %	33.19 %	5.15 %	1.85 %

3. Use Bornhuetter-Ferguson's method to predict the future claim counts. [10 marks]

Observed claim statistics and predicted future development

Accident year	Exposure	Developed to	Observed	pi(cum.)	Theta_(BF)	Outstanding	Ultimate
1	5124	3	156	100 %	3.16E-02	0.0	156.0
2	4719	2	130	98 %	3.16E-02	2.8	132.8
3	3898	1	120	93 %	3.16E-02	8.6	128.6
4	3575	0	84	60 %	3.16E-02	45.4	129.4
Total	17316		490			56.8	546.8

Now we are looking to relax the assumption of one fixed claim rate θ for all accident years.

The Bühlmann-Straub model assumes that every accident year has a *random* claim rate Θ_j. The claim counts N_{jd} are conditionally independent, given Θ_j. The conditional distribution of N_{jd}, given Θ_j is $\text{Poisson}\left(p_j \Theta_j \pi_d\right)$.

4. Derive the functions $b(\Theta_j)$ and $v(\Theta_j)$ of the Bühlmann-Straub model, and show that $b(\Theta_j) = v(\Theta_j)$ in the Poisson case. [20 marks]

$E(N_{jd} \mid \Theta_j) = p_j b(\Theta_j)\pi_d$ and $Var(N_{jd} \mid \Theta_j) = p_j v(\Theta_j)\pi_d \Rightarrow b(\Theta_j) = v(\Theta_j) = \Theta_j$

We denote the mean and variance of Θ_j by $\tau = \mathrm{E}(\Theta_j)$ and $\lambda = \mathrm{Var}(\Theta_j)$.

Given an estimator $\overline{\Theta}_j$ of Θ_j, we predict the number of future claims by $\overline{N}_{j,>J-j} = p_j \overline{\Theta}_j \pi_{>J-j}$.

5. Prove that the mean squared error of $\overline{N}_{j,>J-j}$ is

$$\mathrm{MSE}\left(\overline{N}_{j,>J-j}\right) = \mathrm{E}\left(\overline{N}_{j,>J-j} - N_{j,>J-j}\right)^2 = \left(p_j \pi_{>J-j}\right)^2 \mathrm{E}\left(\overline{\Theta}_j - \Theta_j\right)^2 + p_j \pi_{>J-j} \tau$$

[15 marks]

$$\begin{aligned}
&\mathrm{E}(\overline{N}_{jd} - N_{jd})^2 = \mathrm{E}(p_j \overline{\Theta}_j \pi_d - N_{jd})^2 = \mathrm{EE}\left((p_j \overline{\Theta}_j \pi_d - N_{jd})^2 \middle| \Theta_j\right) \\
&= \mathrm{EE}\left((p_j \overline{\Theta}_j \pi_d - p_j \Theta_j \pi_d + p_j \Theta_j \pi_d - N_{jd})^2 \middle| \Theta_j\right) \\
&= \mathrm{EE}\left((p_j \overline{\Theta}_j \pi_d - p_j \Theta_j \pi_d)^2 \middle| \Theta_j\right) + \mathrm{EE}\left((p_j \Theta_j \pi_d - N_{jd})^2 \middle| \Theta_j\right) + 2\mathrm{EE}\left((p_j \overline{\Theta}_j \pi_d - p_j \Theta_j \pi_d)(p_j \Theta_j \pi_d - N_{jd})^2 \middle| \Theta_j\right) \\
&= \left(p_j \pi_d\right)^2 \mathrm{EE}\left((\overline{\Theta}_j - \Theta_j)^2 \middle| \Theta_j\right) + \mathrm{EVar}\left(N_{jd} \middle| \Theta_j\right) + 0 \text{ (last term zero because the past is cond. indep. of the future)} \\
&= \left(p_j \pi_d\right)^2 \mathrm{E}(\overline{\Theta}_j - \Theta_j)^2 + \mathrm{E}\left(p_j \Theta_j \pi_d\right) \\
&= (p_j \pi_d)^2 Q(z_j) + p_j \pi_d \tau
\end{aligned}$$

Now define $\Theta_j^* = \dfrac{N_{j,\le J-j}}{p_j \pi_{\le J-j}}$.

6. Prove that $\mathrm{E}\left(\Theta_j^* \mid \Theta_j\right) = \Theta_j$ and $\mathrm{Var}\left(\Theta_j^* \mid \Theta_j\right) = \dfrac{\Theta_j}{p_j \pi_{\le J-j}}$. [10 marks]

We assumed that the conditional distribution of N_{jd}, given Θ_j is $\text{Poisson}\left(p_j \Theta_j \pi_d\right)$, and that they are independent. So the conditional distribution of $N_{j,\le J-j}$, given Θ_j is $\text{Poisson}\left(p_j \Theta_j \pi_{\le J-j}\right)$. Thus

$\mathrm{E}\left(\Theta_j^* \mid \Theta_j\right) = \mathrm{E}\left(\dfrac{N_{j,\le J-j}}{p_j \pi_{\le J-j}} \mid \Theta_j\right) = \dfrac{p_j \Theta_j \pi_{\le J-j}}{p_j \pi_{\le J-j}} = \Theta_j$ and

$\mathrm{Var}\left(\Theta_j^* \mid \Theta_j\right) = \mathrm{Var}\left(\dfrac{N_{j,\le J-j}}{p_j \pi_{\le J-j}} \mid \Theta_j\right) = \dfrac{p_j \Theta_j \pi_{\le J-j}}{\left(p_j \pi_{\le J-j}\right)^2} = \dfrac{\Theta_j}{p_j \pi_{\le J-j}}$.

For a non-random credibility factor z, define a credibility estimator of Θ_j by $\overline{\Theta}_j = z\Theta_j^* + (1-z)\tau$.

7. Prove that the mean squared error of $\overline{\Theta}_j$ is

$$\mathrm{MSE}\left(\overline{\Theta}_j\right) = \mathrm{E}\left(\overline{\Theta}_j - \Theta_j\right)^2 = z^2 \frac{\tau}{p_j \pi_{\le J-j}} + (1-z)^2 \lambda.$$ [20 marks]

$$Q(z) = \mathrm{E}\left(\overline{\Theta}_j - \Theta_j\right)^2 = \mathrm{E}\left(z\left(\frac{N_{j,\le J-j}}{p_j \pi_{\le J-j}}\right) + (1-z)\tau - \Theta_j\right)^2 = \mathrm{E}\left(z\left(\left(\frac{N_{j,\le J-j}}{p_j \pi_{\le J-j}}\right) - \Theta_j\right) + (1-z)\left(\tau - \Theta_j\right)\right)^2$$

$$= z^2 \mathrm{E}\left(\left(\frac{N_{j,\le J-j}}{p_j \pi_{\le J-j}}\right) - \Theta_j\right)^2 + (1-z)^2 \mathrm{E}\left(\tau - \Theta_j\right)^2 + 2z(1-z)\mathrm{E}\left(\left(\left(\frac{N_{j,\le J-j}}{p_j \pi_{\le J-j}}\right) - \Theta_j\right)\left(\tau - \Theta_j\right)\right)$$

$$= z^2 \mathrm{E}\mathrm{E}\left(\left(\left(\frac{N_{j,\le J-j}}{p_j \pi_{\le J-j}}\right) - \Theta_j\right)^2 \middle| \Theta_j\right) + (1-z)^2 \lambda + 2z(1-z)\mathrm{E}\mathrm{E}\left(\left(\left(\frac{N_{j,\le J-j}}{p_j \pi_{\le J-j}}\right) - \Theta_j\right)\left(\tau - \Theta_j\right) \middle| \Theta_j\right)$$

$$= z^2 \mathrm{E}\mathrm{Var}\left(\frac{N_{j,\le J-j}}{p_j \pi_{\le J-j}} \middle| \Theta_j\right) + (1-z)^2 \lambda + 0$$

$$= z^2 \mathrm{E}\mathrm{Var}\left(\frac{N_{j,\le J-j}}{p_j \pi_{\le J-j}} \middle| \Theta_j\right) + (1-z)^2 \lambda$$

$$= z^2 \frac{\tau}{p_j \pi_{\le J-j}} + (1-z)^2 \lambda$$

8. Prove that the value of z that minimises the mean squared error in (7), is

$$\zeta_j = \frac{p_j \pi_{\le J-j} \lambda}{p_j \pi_{\le J-j} \lambda + \tau}.$$

[10 marks]

Solve $\frac{\partial}{\partial z} Q(z) = 0$:

$$\frac{\partial}{\partial z} Q(z) = \frac{\partial}{\partial z}\left(z^2 \frac{\tau}{p_j \pi_{\le J-j}} + (1-z)^2 \lambda \right) = 2z \frac{\tau}{p_j \pi_{\le J-j}} - 2(1-z)\lambda = 0$$

$$\Rightarrow z = \frac{p_j \pi_{\le J-j} \lambda}{p_j \pi_{\le J-j} \lambda + \tau} =: \zeta_j$$

Now assume that $\tau = 0.03$ and $\lambda = 0.00003$.

9. Complete the following table.

j	p_j	$N_{j,\le J-j}$	$\pi_{\le J-j}$	Θ_j^*	ζ_j	$\overline{\Theta}_j$	$\overline{N}_{j,>J-j}$	$\mathrm{MSE}(\overline{\Theta}_j)$	$\mathrm{MSE}(\overline{N}_{j,>J-j})$
1	5 124	156	100 %						
2	4 719	130	98 %						
3	3 898	120	93 %						
4	3 575	84	60 %						
Total	17 316	490							

[40 marks]

Accident year	Exposure	Observed	pi(cum.)	Theta(CL)	zeta	Theta_bar(j)	Outstanding	MSEP(Q)	MSEP(N>)
1	5 124	156	100 %	3.04 %	84 %	3.04 %	0	4.90E-06	0
2	4 719	130	98 %	2.81 %	82 %	2.84 %	2	5.33E-06	3
3	3 898	120	93 %	3.31 %	78 %	3.24 %	9	6.49E-06	9
4	3 575	84	60 %	3.93 %	68 %	3.63 %	52	9.56E-06	63
Total	17 316	490		3.16 %			64		74

10. Describe the meaning of the following “attachment conditions”:
    A) Claims incurred, also known as Losses occurring
    B) Claims made
    C) Claims manifestation

[15 marks]

A. Losses occurring in the policy period
B. Claims made/reported during the policy period
C. Right to claim became manifest during the policy period

11. A claim can be thought of passing through different states as time goes by: Covered but not Incurred (CBNI), Incurred but not Reported (IBNR), Reported but not Settled (RBNS) and, finally, Settled. Describe each of these states and give reasons for why a claim can remain in those states for a shorter or longer period.

[15 marks]

Tell about …

CBNI for unexpired contracts
IBNR – claim reporting delayed for various reasons, good or bad. Mention IBNR after public holidays
RBNS – claim assessment, handling, time to ascertain permanent disability, legal wrangles etc.
etc

END